

Symmetries and the compatibility condition for the new translational shape invariant potentials

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Abstract

In this letter we study a class of symmetries of the new translational extended shape invariant potentials. It is proved that a generalization of a compatibility condition introduced in a previous article is equivalent to the usual shape invariance condition. We focus on the recent examples of Odake and Sasaki (infinitely many polynomial, continuous l and multi-index rational extensions). As a byproduct, we obtain new relations, to the best of our knowledge, for Laguerre, Jacobi polynomials and (confluent) hypergeometric functions.

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1. Introduction

The list of shape invariant potentials has remained quite the same until 2008. Then, key contributions of Gómez-Ullate et al. led to a quick and strong development of the subject in recent years. The first steps were the possibility of rationally extend shape-invariant potentials (to obtain non shape invariant ones) [1, 2]. Then, the introduction of the so called X_l exceptional Laguerre and Jacobi polynomials [3, 4] fostered all subsequent works. By the one hand, Quesne (and coworkers) [5, 6, 7] introduced the first examples of rationally extended shape invariant potentials. This idea has been greatly developed by Odake and Sasaki [8, 9, 10, 11, 12] to infinitely many families of rationally extended shape-invariant potentials, even with functions depending on continuous index l and multi-indexed polynomials. They have also extended these ideas to the context of discrete quantum mechanics

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(see, e.g., [13] and references therein). Other works by Grandati [14, 15, 16] have a close relation with the ones cited.

On the other hand, the works [5, 17, 18] inspired our recent article [19], where a compatibility condition has been found that is satisfied by the new examples. Even we have shown that such a condition forces the shape invariance of the examples treated there. It is worth mentioning that [17, 18] are preceded by [20]. This paper continues on the same line of study and shows that the examples of [8, 9, 10, 11] fit perfectly in our framework, satisfying the mentioned compatibility condition.

The letter is organized as follows. In the second section we recall the equations which satisfy the new translational shape invariant potentials of [5, 6, 7, 18, 19]. We prove the equivalence between a generalization of the cited compatibility condition and the usual shape invariance condition. Afterwards, we comment on the isospectrality properties of the potentials involved. In the third section we describe how the examples of [8, 9, 10, 11] fit into our framework. We obtain as a byproduct new relations, to the best of our knowledge, for Laguerre, Jacobi polynomials and (confluent) hypergeometric functions. In the fourth and last section we offer some conclusions.

2. Symmetries and the relation of the compatibility condition with the shape invariance condition

For a brief account of shape invariance, see, e.g., [19] and references therein. In the examples of [5, 6, 7, 8, 9, 10, 11, 18, 19] the superpotential function takes the form of

$$W(x, a) = W_0(x, a) + W_{1+}(x, a) - W_{1-}(x, a), \quad (1)$$

where a denotes the set of parameters under transformation. $W_0(x, a)$ is the superpotential of a pair of shape invariant partner potentials of the classical type. $W_{1+}(x, a)$, $W_{1-}(x, a)$ are logarithmic derivatives which moreover satisfy

$$W_{1-}(x, a) = W_{1+}(x, f(a)), \quad (2)$$

where $f(a)$ in those cases is a translation of a .

The corresponding partner potentials for (1) are

$$\begin{aligned} V(x, a) = & W_0^2(x, a) - W_0'(x, a) \\ & + W_{1+}^2(x, a) + W_{1+}'(x, a) + W_{1-}^2(x, a) + W_{1-}'(x, a) \end{aligned}$$

$$\begin{aligned}
& -2W_0(x, a)W_{1-}(x, a) + 2W_0(x, a)W_{1+}(x, a) \\
& -2W_{1-}(x, a)W_{1+}(x, a) - 2W'_{1+}(x, a) \\
\tilde{V}(x, a) = & W_0^2(x, a) + W'_0(x, a) \\
& + W_{1+}^2(x, a) + W'_{1+}(x, a) + W_{1-}^2(x, a) + W'_{1-}(x, a) \\
& -2W_0(x, a)W_{1-}(x, a) + 2W_0(x, a)W_{1+}(x, a) \\
& -2W_{1-}(x, a)W_{1+}(x, a) - 2W'_{1-}(x, a)
\end{aligned} \tag{3}$$

However, for the examples of [5, 6, 7, 18] such partner potentials reduce to

$$V(x, a) = V_0(x, a) - 2W'_{1+}(x, a), \tag{5}$$

$$\tilde{V}(x, a) = \tilde{V}_0(x, a) - 2W'_{1-}(x, a), \tag{6}$$

where $V_0(x, a)$, $\tilde{V}_0(x, a)$ conform the pair of shape invariant partner potentials associated to $W_0(x, a)$. Thus, it is in principle necessary that the following *compatibility condition* holds:

$$W_{1+}^2 + W'_{1+} + W_{1-}^2 + W'_{1-} - 2W_0W_{1-} + 2W_0W_{1+} - 2W_{1-}W_{1+} = 0 \tag{7}$$

(the dependence on the arguments has been omitted for brevity). Such compatibility condition is the main object of our interest here. First we will discuss a kind of symmetries of the problems of type (1), (2), (3), (4). Afterwards we establish the relation between a generalized compatibility condition and the ordinary shape invariance condition.

2.1. Symmetries of the new translational shape invariant potentials

There exist a class of symmetries of superpotentials of type (1) which satisfy the condition (2) given by the transformations

$$W_{1+}(x, a) = U_{1+}(x, a) - g(x) \tag{8}$$

$$W_{1-}(x, a) = U_{1-}(x, a) - g(x) \tag{9}$$

where $g(x)$ is a function *depending only on* x . The function $g(x)$ must be differentiable in the domain of interest but otherwise arbitrary. For example, $g(x)$ could be any polynomial, e^x , etc. Thus we have

$$W(x, a) = W_0(x, a) + W_{1+}(x, a) - W_{1-}(x, a) = W_0(x, a) + U_{1+}(x, a) - U_{1-}(x, a)$$

The corresponding partner potentials (3), (4) are likewise invariant under (8) and (9). However, their different terms do vary, in such a way that their variations cancel out. Firstly, we have

$$\begin{aligned} & W_{1+}^2 + W_{1+}' + W_{1-}^2 + W_{1-}' - 2W_0W_{1-} + 2W_0W_{1+} - 2W_{1-}W_{1+} \\ &= U_{1+}^2 + U_{1+}' + U_{1-}^2 + U_{1-}' - 2W_0U_{1-} + 2W_0U_{1+} - 2U_{1-}U_{1+} \\ &\quad - 2g'(x) \end{aligned}$$

and moreover

$$\begin{aligned} -2W_{1+}'(x, a) &= -2U_{1+}'(x, a) + 2g'(x) \\ -2W_{1-}'(x, a) &= -2U_{1-}'(x, a) + 2g'(x) \end{aligned}$$

Therefore, if (7) holds, we have

$$\begin{aligned} & U_{1+}^2(x, a) + U_{1+}'(x, a) + U_{1-}^2(x, a) + U_{1-}'(x, a) \\ & - 2W_0(x, a)U_{1-}(x, a) + 2W_0(x, a)U_{1+}(x, a) - 2U_{1-}(x, a)U_{1+}(x, a) = 2g'(x) \end{aligned}$$

This means that by virtue of a symmetry of the problem, the compatibility condition (7) should be generalized in such a way that its right hand side could be a function of x not necessarily equal to zero. This observation leads to our main result in the following subsection.

2.2. Compatibility and shape invariance conditions

For the class of problems described in this letter, there is an equivalence between the mentioned generalized compatibility condition and the usual shape invariance condition, as described in the next Theorem.

Theorem 1. *Assume we have a superpotential of the type*

$$W(x, a) = W_0(x, a) + W_{1+}(x, a) - W_{1-}(x, a), \quad (10)$$

where

$$W_{1-}(x, a) = W_{1+}(x, f(a)),$$

$f(a)$ being the transformation on the parameters a , and $W_0(x, a)$ satisfies the shape invariance condition

$$W_0^2(x, a) - W_0^2(x, f(a)) + W_0'(x, f(a)) + W_0'(x, a) = R(f(a)). \quad (11)$$

Then, the shape invariant condition for $W(x, a)$

$$W^2(x, a) - W^2(x, f(a)) + W'(x, f(a)) + W'(x, a) = R(f(a)) \quad (12)$$

holds if and only if

$$\begin{aligned} & W_{1+}^2(x, a) + W_{1+}'(x, a) + W_{1-}^2(x, a) + W_{1-}'(x, a) \\ & - 2W_0(x, a)W_{1-}(x, a) + 2W_0(x, a)W_{1+}(x, a) - 2W_{1-}(x, a)W_{1+}(x, a) \\ & = \epsilon(x) \end{aligned} \quad (13)$$

for some non-singular function $\epsilon(x)$ of x only.

Proof

The condition of shape invariance (12) reads in this case

$$\begin{aligned} & W^2(x, a) - W^2(x, f(a)) + W'(x, f(a)) + W'(x, a) - R(f(a)) = \\ & W_0^2(x, a) - W_0^2(x, f(a)) + W_0'(x, f(a)) + W_0'(x, a) - R(f(a)) \\ & + W_{1+}^2(x, a) + W_{1+}'(x, a) + W_{1-}^2(x, a) + W_{1-}'(x, a) \\ & - 2W_0(x, a)W_{1-}(x, a) + 2W_0(x, a)W_{1+}(x, a) - 2W_{1-}(x, a)W_{1+}(x, a) \\ & - [W_{1+}^2(x, f(a)) + W_{1+}'(x, f(a)) + W_{1-}^2(x, f(a)) + W_{1-}'(x, f(a)) \\ & - 2W_0(x, f(a))W_{1-}(x, f(a)) + 2W_0(x, f(a))W_{1+}(x, f(a)) \\ & - 2W_{1-}(x, f(a))W_{1+}(x, f(a))] - 2W_{1-}'(x, a) + 2W_{1+}'(x, f(a)) = 0 \end{aligned} \quad (14)$$

With the hypothesis that $W_0(x, a)$ satisfies (11), also that $W_{1-}(x, a) = W_{1+}(x, f(a))$ and that

$$\begin{aligned} & W_{1+}^2(x, a) + W_{1+}'(x, a) + W_{1-}^2(x, a) + W_{1-}'(x, a) \\ & - 2W_0(x, a)W_{1-}(x, a) + 2W_0(x, a)W_{1+}(x, a) - 2W_{1-}(x, a)W_{1+}(x, a) \\ & = \epsilon(x) \\ & W_{1+}^2(x, f(a)) + W_{1+}'(x, f(a)) + W_{1-}^2(x, f(a)) + W_{1-}'(x, f(a)) \\ & - 2W_0(x, f(a))W_{1-}(x, f(a)) + 2W_0(x, f(a))W_{1+}(x, f(a)) \\ & - 2W_{1-}(x, f(a))W_{1+}(x, f(a)) \\ & = \epsilon(x) \end{aligned}$$

the shape invariance condition is readily satisfied.

Conversely, with the above hypothesis we assume that the shape invariance condition (12) is satisfied, therefore (14) is also satisfied. Taking into account (11) and $W_{1-}(x, a) = W_{1+}(x, f(a))$ and rearranging, (14) becomes

$$\begin{aligned}
& W_{1+}^2(x, a) + W'_{1+}(x, a) + W_{1-}^2(x, a) + W'_{1-}(x, a) \\
& - 2W_0(x, a)W_{1-}(x, a) + 2W_0(x, a)W_{1+}(x, a) \\
& - 2W_{1-}(x, a)W_{1+}(x, a) = \\
& W_{1+}^2(x, f(a)) + W'_{1+}(x, f(a)) + W_{1-}^2(x, f(a)) + W'_{1-}(x, f(a)) \\
& - 2W_0(x, f(a))W_{1-}(x, f(a)) + 2W_0(x, f(a))W_{1+}(x, f(a)) \\
& - 2W_{1-}(x, f(a))W_{1+}(x, f(a))
\end{aligned}$$

that is, the expression evaluated at (x, a) equals the expression itself evaluated at $(x, f(a))$, thus both expressions must be equal to a function of x only, namely, $\epsilon(x)$. This ends the proof of the Theorem.

Remarks

1. In actual examples it is observed that (13) is satisfied with $\epsilon(x) = 0$, which is a slightly stronger condition that in particular implies shape invariance.

2. Note that Ho proposes in [21, 22] a similar form to the superpotential (1), but that approach is different: other relations, different from (7) or (13) are satisfied. As an example of this, in [19] it is shown that the harmonic oscillator and the Morse potential admit no non-trivial extensions by our means. However with the technique of Ho they do. See also [23, 24, 25, 26].

3. We observe that the potentials in (5) and (6) are related by construction by a first order intertwining relation as described in Section 2 of [19] with superpotential (1). The fulfillment of condition (7) provides a cancelation of some of their terms (it is the same condition for $V(x, a)$ in (5) and $\tilde{V}(x, a)$ in (6)). Thus the isospectrality (maybe up to the ground state of one of them) of the potentials (5) and (6) is ensured. See [27, 28] for a group theoretical explanation of the intertwining technique.

4. Another question is the isospectrality of the mentioned potentials with the ordinary shape invariant potentials $V_0(x, a)$ and $\tilde{V}_0(x, a)$. This is also easy to justify: with the conditions (2) and (13) the shape invariant relation (12) for the potentials of (5) and (6) becomes identical to that of the partner potentials $V_0(x, a)$ and $\tilde{V}_0(x, a)$. In particular, the quantity $R(f(a))$, from which the spectrum of the potentials is calculated, is identical in both cases, showing the mentioned isospectrality (maybe up to the ground state of one of them). See also [7, 11] for an approximation to such an isospectrality based on the intertwining technique.

5. We have established that the generalized compatibility condition (13) is equivalent to the ordinary shape invariance condition (12) in the mentioned circumstances. However, the former condition is simpler to work with than the latter for the cases studied in [5, 6, 7, 18, 19] and in this letter.

6. The condition (7) has been shown, for W' s satisfying the Bernoulli equation $W' + W^2 - k_1(x)W = 0$ (where $k_1(x) = c \coth(cx)$, etc.), in the examples of [19], to imply (2) in particular. This means that such conditions are not really independent in specific examples.

The compatibility condition (13) admits another, even simpler, form. Denoting

$$W_{1+}(x, a) = \frac{\psi'_{1+}(x, a)}{\psi_{1+}(x, a)}, \quad W_{1-}(x, a) = \frac{\psi'_{1-}(x, a)}{\psi_{1-}(x, a)}$$

it becomes

$$\frac{1}{\psi_{1+}}(\psi''_{1+} + 2W_0\psi'_{1+}) + \frac{1}{\psi_{1-}}(\psi''_{1-} - 2W_0\psi'_{1-}) - 2\frac{\psi'_{1+}\psi'_{1-}}{\psi_{1+}\psi_{1-}} = \epsilon(x) \quad (15)$$

where the dependence on the arguments (x, a) has been dropped for simplicity. For the case of $\epsilon(x) = 0$ it follows

$$(\psi''_{1+} + 2W_0\psi'_{1+})\psi_{1-} + (\psi''_{1-} - 2W_0\psi'_{1-})\psi_{1+} - 2\psi'_{1+}\psi'_{1-} = 0 \quad (16)$$

In terms of the functions $\psi_{1+}(x, a)$, $\psi_{1-}(x, a)$, the symmetries of Subsection 2.1 are expressed in the following way. The functions change as

$$\begin{aligned} \psi_{1+}(x, a) &= \exp\left(-\int g(x) dx\right) \chi_{1+}(x, a) \\ \psi_{1-}(x, a) &= \exp\left(-\int g(x) dx\right) \chi_{1-}(x, a) \end{aligned}$$

where $U_{1+}(x, a) = \frac{\chi'_{1+}(x, a)}{\chi_{1+}(x, a)}$ and $U_{1-}(x, a) = \frac{\chi'_{1-}(x, a)}{\chi_{1-}(x, a)}$.

3. Examples

In this section we study the fulfillment of the compatibility condition (16) for the examples of [8, 9, 10, 11]. These cases are specially well suited for our purposes, since they take the form of Section 2 and are known to be shape invariant. By the symmetry property of these problems, it suffices to study the compatibility condition (16), which will be obtained directly in all cases. We will obtain as a byproduct new relations, to the best of our knowledge, of Laguerre, Jacobi polynomials and (confluent) hypergeometric functions.

3.1. Polynomial shape invariant extensions of the radial oscillator and Darboux–Pöschl–Teller potentials

3.1.1. Radial oscillator

According to [8, 9, 10], the extended partner potentials of the radial oscillator have a superpotential of the form

$$W_l(x, g) = W_0(x, g + l) + \frac{\xi'_l(x^2, g + 1)}{\xi_l(x^2, g + 1)} - \frac{\xi'_l(x^2, g)}{\xi_l(x^2, g)}$$

where $x > 0$,

$$\begin{aligned} W_0(x, g) &= -x + \frac{g}{x} \\ \xi_l(x, g) &= L_l^{(g+l-\frac{3}{2})}(-x) \end{aligned}$$

and $L_n^{(a)}(x)$ are Laguerre polynomials.

We will try to check (16) directly choosing (with a slight abuse of notation)

$$\begin{aligned} W_0(x, a) &= W_0(x, g + l) \\ \psi_{1+}(x, a) &= \xi_l(x^2, g + 1) \\ \psi_{1-}(x, a) &= \xi_l(x^2, g) \end{aligned}$$

and by writing it in another way, using the relation (2.41) of [10], namely (dependence on arguments dropped)

$$\psi_{1+}'' = 4l\psi_{1+} - 2\left(\frac{g+l}{x} + x\right)\psi_{1+}' \quad (17)$$

$$\psi_{1-}'' = 4l\psi_{1-} - 2\left(\frac{g+l-1}{x} + x\right)\psi_{1-}' \quad (18)$$

Thus the relation (16) becomes

$$8l\psi_{1+}\psi_{1-} + \frac{2(1-2g-2l)}{x}\psi_{1+}\psi_{1-}' - 4x\psi_{1-}\psi_{1+}' - 2\psi_{1-}'\psi_{1+}' = 0 \quad (19)$$

This last relation can be proved using the equations (3.5) and (3.6) of [10]. That implies, in particular, the fulfillment of (12) for this case. In [10] it is proved (12) directly for the current case.

The relations (16) and (19) are new, and equivalent to each other, for Laguerre polynomials.

3.1.2. Trigonometric Darboux-Pöschl-Teller potential

According to [8, 9, 10], the extended partner potentials of the trigonometric Darboux-Pöschl-Teller potential have a superpotential of the form

$$W_l(x, g, h) = W_0(x, g + l, h + l) + \frac{\xi'_l(\cos(2x), g + 1, h + 1)}{\xi_l(\cos(2x), g + 1, h + 1)} - \frac{\xi'_l(\cos(2x), g, h)}{\xi_l(\cos(2x), g, h)}$$

where $x \in (0, \frac{\pi}{2})$,

$$\begin{aligned} W_0(x, g, h) &= g \cot(x) - h \tan(x) \\ \xi_l(x, g, h) &= P_l^{(-g-l-\frac{1}{2}, h+l-\frac{3}{2})}(x) \end{aligned}$$

and $P_n^{(a,b)}(x)$ are Jacobi polynomials.

We will try to check (16) by choosing (with a slight abuse of the notation)

$$\begin{aligned} W_0(x, a) &= W_0(x, g + l, h + l) \\ \psi_{1+}(x, a) &= \xi_l(\cos(2x), g + 1, h + 1) \\ \psi_{1-}(x, a) &= \xi_l(\cos(2x), g, h) \end{aligned}$$

Moreover, we transform (16) by using the relation (2.41) of [10], namely

$$\begin{aligned} \psi''_{1+} &= 4l(g - h - l + 1)\psi_{1+} + 2((g + l + 1)\cot x + (h + l)\tan x)\psi'_{1+} \\ \psi''_{1-} &= 4l(g - h - l + 1)\psi_{1-} + 2((g + l)\cot x + (h + l - 1)\tan x)\psi'_{1-} \end{aligned}$$

Thus, the relation (16) becomes

$$\begin{aligned} -8l(h - g + l - 1)\psi_{1+}\psi_{1-} + 2(2h + 2l - 1)\tan x \psi_{1+}\psi'_{1-} \\ + 2(2g + 2l + 1)\cot x \psi_{1-}\psi'_{1+} - 2\psi'_{1-}\psi'_{1+} = 0 \end{aligned} \quad (20)$$

Such relation can be proved directly using (3.12) and (3.13) of [10]. In such a paper, it has been proved the shape invariance condition (12) directly. We have proved it checking that the stronger (and simpler) condition (16) or (20) holds.

For this case, (16) and (20) are new relations, equivalent to each other, for Jacobi polynomials.

3.1.3. Hyperbolic Darboux-Pöschl-Teller potential

According to [8, 9, 10], the extended partner potentials of the hyperbolic Darboux-Pöschl-Teller potential have a superpotential of the form

$$W_l(x, g, h) = W_0(x, g + l, h - l) + \frac{\xi'_l(\cosh(2x), g + 1, h - 1)}{\xi_l(\cosh(2x), g + 1, h - 1)} - \frac{\xi'_l(\cosh(2x), g, h)}{\xi_l(\cosh(2x), g, h)}$$

where $x > 0$,

$$\begin{aligned} W_0(x, g, h) &= g \coth(x) - h \tanh(x) \\ \xi_l(x, g, h) &= P_l^{(-g-l-\frac{1}{2}, -h+l-\frac{3}{2})}(x) \end{aligned}$$

and $P_n^{(a,b)}(x)$ are again Jacobi polynomials.

We will try to check (16) by choosing

$$\begin{aligned} W_0(x, a) &= W_0(x, g+l, h-l) \\ \psi_{1+}(x, a) &= \xi_l(\cosh(2x), g+1, h-1) \\ \psi_{1-}(x, a) &= \xi_l(\cosh(2x), g, h) \end{aligned}$$

and transforming the cited condition by using the relation (2.41) of [10], namely

$$\begin{aligned} \psi_{1+}'' &= 4l(l-g-h-1)\psi_{1+} + 2((g+l+1)\coth x + (h-l)\tanh x)\psi_{1+}' \\ \psi_{1-}'' &= 4l(l-g-h-1)\psi_{1-} + 2((g+l)\coth x + (h-l+1)\tanh x)\psi_{1-}' \end{aligned}$$

Thus the relation (16) becomes

$$\begin{aligned} &-8l(h+g-l+1)\psi_{1+}\psi_{1-} + 2(1+2h-2l)\tanh x \psi_{1+}\psi_{1-}' \\ &+ 2(1+2g+2l)\coth x \psi_{1-}\psi_{1+}' - 2\psi_{1-}'\psi_{1+}' = 0 \end{aligned} \quad (21)$$

This last relation can be proved by using equations (3.12) and (3.13) of [10], as in the previous case. Therefore, the shape invariance for this case holds in particular. In [10], the relation (12) has been proved directly.

For this case, (16) and (21) are new relations, equivalent to each other, for Jacobi polynomials.

3.2. Continuous l shape invariant extensions of the radial oscillator and trigonometric Darboux–Pöschl–Teller potentials

3.2.1. Radial oscillator

According to [11], the extended partner potentials of the radial oscillator with continuous $l > 0$ have a superpotential of the form

$$W_l(x, g) = W_0(x, g+l) + \frac{\xi_l'(x^2, g+1)}{\xi_l(x^2, g+1)} - \frac{\xi_l'(x^2, g)}{\xi_l(x^2, g)}$$

where $x > 0$,

$$\begin{aligned} W_0(x, g) &= -x + \frac{g}{x} \\ \xi_l(x, g) &= \frac{\Gamma(g + 2l - \frac{1}{2})}{\Gamma(l + 1)\Gamma(g + l - \frac{1}{2})} {}_1F_1 \left(\begin{matrix} -l \\ g + l - \frac{1}{2} \end{matrix} \middle| -x \right) \end{aligned}$$

and ${}_1F_1 \left(\begin{matrix} a \\ b \end{matrix} \middle| x \right)$, $\Gamma(x)$ are the confluent hypergeometric and Gamma functions, respectively.

We choose (with a slight abuse of notation)

$$\begin{aligned} W_0(x, a) &= W_0(x, g + l) \\ \psi_{1+}(x, a) &= \xi_l(x^2, g + 1) \\ \psi_{1-}(x, a) &= \xi_l(x^2, g) \end{aligned}$$

in order to check whether (16) is satisfied. We first transform it by using the relation (3.9) of [11], namely (17) and (18). Therefore, (16) is transformed into (19) again. Such relation can be proved again for the current ψ_{1+}, ψ_{1-} by using properties (3.10) and (3.11) of [11]. Thus the compatibility condition holds and as a result also the shape invariance condition does. This last result has been obtained directly in [11].

For this case, (16) and (19) are new relations, equivalent to each other, for confluent hypergeometric functions.

3.2.2. Trigonometric Darboux-Pöschl-Teller potential

According to [11], the extended partner potentials of the trigonometric Darboux-Pöschl-Teller potential with continuous $l > 0$ have a superpotential of the form

$$W_l(x, g, h) = W_0(x, g + l, h + l) + \frac{\xi'_l(\cos(2x), g + 1, h + 1)}{\xi_l(\cos(2x), g + 1, h + 1)} - \frac{\xi'_l(\cos(2x), g, h)}{\xi_l(\cos(2x), g, h)}$$

where $x \in (0, \frac{\pi}{2})$,

$$\begin{aligned} W_0(x, g, h) &= g \cot(x) - h \tan(x) \\ \xi_l(x, g, h) &= \frac{\Gamma(g + 2l - \frac{1}{2})}{\Gamma(l + 1)\Gamma(g + l - \frac{1}{2})} {}_2F_1 \left(\begin{matrix} -l, g - h + l - 1 \\ g + l - \frac{1}{2} \end{matrix} \middle| \frac{1 - x}{2} \right) \end{aligned}$$

and ${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right)$ is the hypergeometric function.

We denote again

$$\begin{aligned} W_0(x, a) &= W_0(x, g + l, h + l) \\ \psi_{1+}(x, a) &= \xi_l(\cos(2x), g + 1, h + 1) \\ \psi_{1-}(x, a) &= \xi_l(\cos(2x), g, h) \end{aligned}$$

in order to check that (16) holds. We transform it first using the result (3.9) of [11], namely

$$\begin{aligned} \psi_{1+}'' &= -4l(g - h + l - 1)\psi_{1+} \\ &\quad - 2((g + h + 2l + 1) \csc(2x) + (g - h - 1) \cot(2x)) \psi_{1+}' \\ \psi_{1-}'' &= -4l(g - h + l - 1)\psi_{1-} \\ &\quad - 2((g + h + 2l - 1) \csc(2x) + (g - h - 1) \cot(2x)) \psi_{1-}' \end{aligned}$$

Thus the relation (16) becomes

$$\begin{aligned} &-8l(g - h + l - 1)\psi_{1+}\psi_{1-} - 2(2g + 2l - 1) \cot x \psi_{1+}\psi_{1-}' \\ &- 2(2h + 2l + 1) \tan x \psi_{1-}\psi_{1+}' - 2\psi_{1-}'\psi_{1+}' = 0 \end{aligned} \quad (22)$$

This last equation can be proved directly using the equations (3.10) and (3.11) of [11], thus fulfilling the compatibility condition. As a consequence, (12) holds (something which has been checked directly in [11]).

For this case, (16) and (22) are new relations amongst hypergeometric functions equivalent to each other.

4. Conclusions and outlook

We have studied the fulfillment of the compatibility condition introduced in [19] in the cases of the extended shape invariant potentials of [8, 9, 10, 11]. Firstly, we have proved that for the form of the superpotential (1), where $W_0(x, a)$ generates a pair of shape invariant potentials of the classical type and the extra terms satisfy (2), the compatibility condition (13) is equivalent to the ordinary shape invariance condition for the full superpotential (12). Then, the cited examples are exactly of the form described in Section 2. We check directly whether the compatibility condition (16) holds and indeed we prove it in all cases, using previous results of [10, 11]. Thus, for the cases studied we provide an alternative and simpler way of proving shape invariance.

The multi-index polynomial extensions to the radial oscillator and trigonometric Darboux-Pöschl-Teller potentials introduced in [12] are shown to be shape

invariant and they are of the form described in Section 2, thus the compatibility condition (13) must hold in that cases as well.

It would be interesting to see whether there exists non-trivial rational extensions to other shape invariant potentials of the Infeld and Hull classification [29, 30, 31, 32] (with superpotential of the type $k_0(x) + mk_1(x)$) to infinitely many polynomial and continuous l functions analogous to that of [8, 9, 10, 11]. If these examples do exist, the relation (13) must hold again.

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